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2009 J. Phys. A: Math. Theor. 42 285309

(http://iopscience.iop.org/1751-8121/42/28/285309)

IOP Publishing Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **42** (2009) 285309 (9pp) doi:10.1088/1751-8113/42/28/285309

Generalizations of the pseudospin operator to test the Bell inequality for the TMSV state

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Received 11 December 2008, in final form 25 May 2009
Published 26 June 2009
Online at stacks.iop.org/JPhysA/42/285309

Abstract

Chen *et al* (2002 *Phys. Rev. Lett.* **88** 040406) proposed a method to measure the entanglement of a bipartite quantum system involving continuous variables. The approach is based upon the construction of a pseudospin operator. An important difference with the spin case is an infinite degeneracy of the eigenkets associated with the pseudospin. In this paper, we analyze the role played by such a degeneracy in the study of the locality of the two-mode-squeezed vacuum state. We construct other representations for the pseudospin and study the corresponding Bell inequalities.

PACS number: 03.65.Ud

1. Introduction

The application of Bell's inequalities [1, 2] to quantum systems involving continuous variables (CVs) has been formulated in terms of the Wigner representation [3], or by using a map between the CVs and a spin-$1/2$ system [4, 5]. In order to formulate the Bell inequalities, the existence of a dichotomic observable [6] with eigenvalues ± 1 is required. Once the observable is chosen the Bell operator is built [7] and the inequality is expressed in terms of its expectation value. The quantum state used to define the expectation value is considered to be the source of entanglement and nonlocality.

The authors in [4] constructed an analogy between the two-spin-$1/2$-particle system and the two-mode-squeezed vacuum (TMSV) state. To do so, they defined the so-called pseudospin operator. Using this analogy, the authors show that the TMSV leads to the violation of the Bell–CHSH inequality and even that it is maximally violated in the infinite squeezing limit. A different mapping between the spin-$1/2$ and the TMSV was introduced in [5]. The author generalizes the pseudospin by constructing a set of operators labeled by a parameter d. The analogy between the spin-$1/2$ and the continuous variable system relies on the fact that pseudospin operators have the same properties as the Pauli matrices. However, we have to realize that remarkable differences exist between the spin-1/2 and the continuous variable; indeed for the spin-1/2 system there is no degeneracy, while for the pseudospin case there is an infinite degeneracy [8].

Our purpose in this paper is twofold. The first question is to determine the role played by the degeneracy, i.e. we ask ourselves if the degeneracy present in the pseudospin approach to the TMSV state is of any relevance in the context of the Bell inequalities. The second question we address is if the violation of the Bell inequalities depends only upon the algebra of the operators involved or if it is representation dependent. In order to answer the first question we quantify the degeneracy; this is achieved by truncating the space of states and then considering the violation of the Bell inequality as a function of the dimensionality of the Hilbert space. To deal with the second question, we construct 'generalized' pseudospin operators and derive the corresponding Bell inequality.

2. The spin-1/2 system and its analogue

2.1. Spin-1/2 and the CHSH inequality

Consider a set with only two elements $\{-1, 1\}$. Given four elements of this set $A, A', B, B' \in \{-1, 1\}$ then

$$-2 \leqslant \langle AB\rangle + \langle A'B\rangle + \langle AB'\rangle - \langle A'B'\rangle \leqslant 2, \tag{1}$$

where the brackets represent an average over a given set. This is the CHSH inequality. If a physical system is locally realistic then it should satisfy this inequality, else it is *not* locally realistic [2].

For example, if we have a system composed of two-spin-1/2 particles, we can take A, A', B and B' as the spin projection of each particle over a given direction, let us say over the axes $\hat{a}, \hat{a}', \hat{b}$ and $\hat{b}'$. Then

$$A = \hat{a}\cdot\vec{\sigma}_1, \qquad A' = \hat{a}'\cdot\vec{\sigma}_1, \qquad B = \hat{b}\cdot\vec{\sigma}_2, \qquad B' = \hat{b}'\cdot\vec{\sigma}_2, \tag{2}$$

where $\vec{\sigma}_i$ is the Pauli matrix associated with the ith particle. The Bell operator $\mathcal{B}$ for the two-qubit system is defined as

$$\mathcal{B} = (\hat{a}\cdot\vec{\sigma}_1)\otimes(\hat{b}\cdot\vec{\sigma}_2) + (\hat{a}\cdot\vec{\sigma}_1)\otimes(\hat{b}'\cdot\vec{\sigma}_2) + (\hat{a}'\cdot\vec{\sigma}_1)\otimes(\hat{b}\cdot\vec{\sigma}_2) - (\hat{a}'\cdot\vec{\sigma}_1)\otimes(\hat{b}'\cdot\vec{\sigma}_2). \tag{3}$$

From (1)–(3), the Bell–CHSH inequality follows

$$|\langle\mathcal{B}\rangle| \leqslant 2. \tag{4}$$

Using the $SU(2)$ properties of the Pauli matrices it is easy to prove that

$$\mathcal{B}^2 = 4I_{2\times 2} + 4(\hat{a}\times\hat{a}'\cdot\vec{\sigma}_1)\otimes(\hat{b}\times\hat{b}'\cdot\vec{\sigma}_2). \tag{5}$$

This relation leads to the Cirlel'son inequality [11]

$$|\langle\mathcal{B}\rangle| \leqslant 2\sqrt{2}. \tag{6}$$

Experiments show that the Bell–CHSH inequality is violated and that the Cirel'son inequality is satisfied [9]; this is interpreted in the sense that, for example, the singlet two-spin-1/2 particle system is not locally realistic.

The following is a summary of the actual quantum mechanical calculation for a two-spin-1/2 system. In order to simplify the calculations, the unitary vectors $\hat{a}, \hat{a}', \hat{b}$ and $\hat{b}'$ are defined by the following values of the spherical coordinates:

$$\phi_a = \phi_{a'} = \phi_b = \phi_{b'} = 0, \tag{7}$$

$$\theta_a = 0, \qquad \theta_{a'} = \pi/2, \qquad \theta_b = -\theta_{b'}. \tag{8}$$

The expectation value of the Bell operator reduces to

$$\langle \mathcal{B} \rangle = 2(\cos\theta_b I + \sin\theta_b F), \tag{9}$$

where

$$I = \langle s_z \otimes s_z \rangle, \qquad F = \langle s_x \otimes s_x \rangle, \tag{10}$$

and the brackets denote the mean value for the quantum state of interest. In terms of I and F the maximum expectation value is

$$\langle \mathcal{B} \rangle_{\max} = 2(\sqrt{I^2 + F^2}). \tag{11}$$

2.2. *Continuous variable system*

In terms of the creation and annihilation operators of each field $a_1, a_1^\dagger, a_2$ and $a_2^\dagger$, the TMSV state can be written as

$$|\mathrm{TMSV}\rangle = \mathrm{e}^{r(a_1^\dagger a_2^\dagger - a_1 a_2)}|00\rangle = \sum_{n=0}^{\infty} \frac{(\tanh r)^n}{\cosh r}|nn\rangle, \tag{12}$$

where $|nn\rangle = |n\rangle \otimes |n\rangle$ and $\{|n\rangle\}, n = 0, 1, 2\ldots$ is the Fock state basis for each field. The TMSV is the continuous variable analogue of the discrete singlet entangled state, and it is a regularized version of the state used by EPR. The EPR state is recovered in the $r \to \infty$ limit [12].

In order to test the locality of the TMSV, the authors in [4] constructed an analogy between the spin-1/2 system and the two-mode electromagnetic field. To this end, they introduced the pseudospin operators

$$s_x = \sum_{n=0}^{\infty} |2n+1\rangle\langle 2n| + |2n\rangle\langle 2n+1|, \tag{13a}$$

$$s_y = i\sum_{n=0}^{\infty} |2n\rangle\langle 2n+1| - |2n+1\rangle\langle 2n|, \tag{13b}$$

$$s_z = \sum_{n=0}^{\infty} |2n+1\rangle\langle 2n+1| - |2n\rangle\langle 2n| \tag{13c}$$

and the corresponding ladder operators

$$2s_\pm = s_x \pm \mathrm{i} s_y. \tag{14}$$

These operators have the same properties as the Pauli matrices. A new Bell operator can be defined just by replacing in (3) the Pauli matrices by the pseudospin operators

$$\mathcal{B}_{\mathrm{ps}} = (\hat{a}\cdot\vec{s}_1)\otimes(\hat{b}\cdot\vec{s}_2) + (\hat{a}\cdot\vec{s}_1)\otimes(\hat{b}'\cdot\vec{s}_2) + (\hat{a}'\cdot\vec{s}_1)\otimes(\hat{b}\cdot\vec{s}_2) - (\hat{a}'\cdot\vec{s}_1)\otimes(\hat{b}'\cdot\vec{s}_2). \tag{15}$$

Thus, in order to evaluate the expectation value of the Bell operator for the TMSV state all we need are the values of I and F for the case under consideration, which turn out to be

$$I = 1, \qquad F = \frac{2T}{1+T^2}, \tag{16}$$

where $T = \tanh(r)$, so that the maximum expectation value of the Bell operator is

$$\langle \mathcal{B}_{\mathrm{ps}} \rangle_{\max} = 2\frac{\sqrt{T^4 + 6T^2 + 1^2}}{1+T^2} = 2\sqrt{1+\tanh^2 2r}. \tag{17}$$

We see that for $r > 0$ the Bell inequality is always violated, and since $0 \leqslant \tanh^2(2r) \leqslant 1$ the Cirel'son inequality is always satisfied. Note that $K = \tanh(2r)$ can be used as a measure of entanglement.

2.2.1. Larson's extension of the pseudospin operator. An alternative analogy to the spin operator was proposed by Larson [5], who constructed generalized pseudospin operators:

$$s_{x,d} = \sum_{n=0}^{\infty}\sum_{k=0}^{d-1} |2dn+k+d\rangle\langle 2dn+k| + |2dn+k\rangle\langle 2dn+k+d|, \tag{18a}$$

$$s_{y,d} = i\sum_{n=0}^{\infty}\sum_{k=0}^{d-1} |2dn+k+d\rangle\langle 2dn+k| - |2dn+k\rangle\langle 2dn+k+d|, \tag{18b}$$

$$s_{z,d} = \sum_{n=0}^{\infty}\sum_{k=0}^{d-1} |2dn+k\rangle\langle 2dn+k| - |2dn+d+k\rangle\langle 2dn+d+k|. \tag{18c}$$

For $d = 1$, the pseudospin operators (13) are recovered. For this generalization, we obtain

$$I = 1, \qquad F = \frac{2T^d}{1+T^{2d}}, \tag{19}$$

and

$$\langle \mathcal{B}_d\rangle_{\max} = 2\sqrt{I^2+F^2} = \frac{2\sqrt{T^{4d}+6T^{2d}+1}}{1+T^{2d}}. \tag{20}$$

The behavior of $\langle \mathcal{B}_d\rangle_{\max}$ as a function of r for different values of d is shown in figure 1(a). It is worth remarking that this is the same result as (17) with the replacement $T \to T^d$. Again in this case, K_d can be considered as a measure of entanglement

$$K_d = F^2 = 4\frac{T^{2d}}{(1+T^{2d})^2}. \tag{21}$$

3. Generalizations

3.1. The role of degeneracy

It is important to note that for the spin-1/2 system there is a unique ket $|+\rangle$ such that $\sigma_z|+\rangle = |+\rangle$, while for the pseudospin operator there is an infinite set of eigenkets of s_z such that $s_z|2k+1\rangle = |2k+1\rangle$, i.e. for the pseudospin operator there is an infinite degeneracy not present for the spin system. Moreover, the two sets of operators (equations (13) and (18)) are defined over infinite-dimensional Hilbert spaces. A way to quantify the degeneracy is to keep only a finite number of terms in the definition of the operators. To this end, we introduce a finite-dimensional Hilbert space generated by the basis $\mathbf{VT} = \{|0,0\rangle,\ldots,|0,N-1\rangle,\ldots,|N-1,0\rangle,\ldots,|N-1,N-1\rangle\}$ with $N = 2d(L+1)$. In terms of these basis vectors, the following complete set of independent operators can be constructed:

$$s'_{x,d} = \sum_{n=0}^{L}\sum_{k=0}^{d-1} |2dn+k+d\rangle\langle 2dn+k| + |2dn+k\rangle\langle 2dn+k+d|, \tag{22a}$$

$$s'_{y,d} = i\sum_{n=0}^{L}\sum_{k=0}^{d-1} |2dn+k+d\rangle\langle 2dn+k| - |2dn+k\rangle\langle 2dn+k+d|, \tag{22b}$$

$$s'_{z,d} = \sum_{n=0}^{L}\sum_{k=0}^{d-1} |2dn+k\rangle\langle 2dn+k| - |2dn+d+k\rangle\langle 2dn+d+k|. \tag{22c}$$

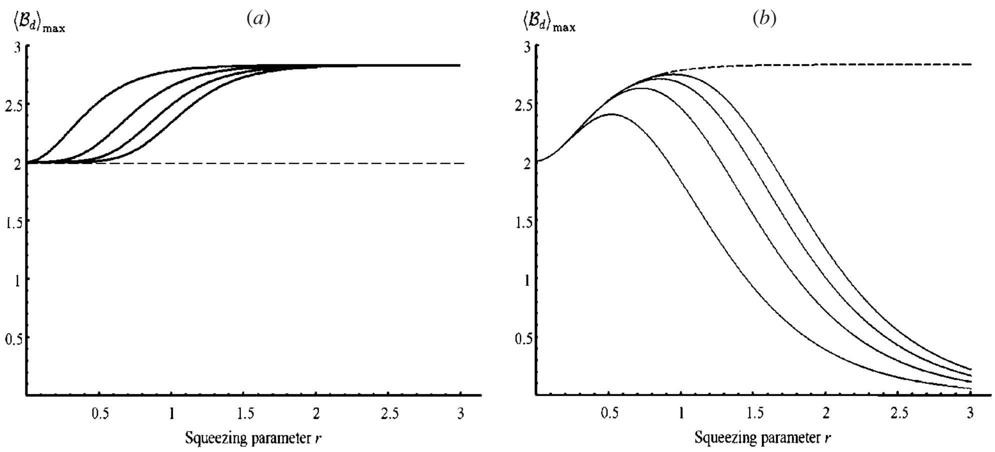

Figure 1. (*a*) The expectation value of Bell's operator built in terms of Larson's $\vec{s}\,'_d$ pseudospin as a function of the squeezing parameter r, between TMSV states, for $d = 1, 2, 3, 4$. The curves appear from left to right as d increases. $d = 1$ corresponds to Chen's pseudospin case, while the dashed line shows the local realistic limit. (*b*) The expectation value of Bell's operator built in terms of the truncated $\vec{s}\,'_d$ operators (equation (22)), between TMSV states, for $d = 1$ and $L = 0, 1, 2, 3$. The curves appear from the left to right as L increases. The case $d = 1$ and $L \to \infty$ are shown with a dashed line. $d = 1$ and $L \to \infty$ correspond to the pseudospin case.

Very much as the pseudospin, the new operator equation (22) satisfies the $SU(2)$ algebra, but now the degeneracy D of the s'_{z_d} operator is finite, $D = d(2L + 1)$. In order to calculate the expectation value, it is necessary to work with properly normalized states [13]:

$$|\mathrm{TMSV}\rangle_{\mathcal{H}_N} = \sqrt{\frac{1-T^2}{1-T^{4d(L+1)}}} \sum_{p=0}^{2d(L+1)-1} T^p |pp\rangle. \tag{23}$$

The I and F expectation values of the s' operator equation (22) between the $|\mathrm{TMSV}\rangle_{\mathcal{H}_N}$ states lead to Chen's results (equations (16) and (17)). That is I, F and, therefore, the vacuum expectation value of the Bell operator do not depend on L, i.e. the violation of the Bell inequality is degeneracy independent.

An alternative is to take the vacuum expectation value of the s' operator equation (22) not between states belonging to the **VT** basis but between the TMSV states. In this case, we obtain

$$I = 1 - T^{4d(L+1)}, \qquad F = (1 - T^{4d(L+1)})\frac{2T^d}{1+T^{2d}}, \tag{24}$$

and the expectation value of the Bell operator is

$$\langle \mathcal{B} \rangle = \frac{2(1 - T^{4d(L+1)})\sqrt{6T^{2d} + T^{4d} + 1}}{(1 + T^{2d})}, \tag{25}$$

i.e. the vacuum expectation value of the Bell inequality depends, through L, on the degeneracy of the operators as shown in figure 1(*b*). However, the interpretation of this result is not trivial because the operator s' ceases to be dichotomic since besides ± 1 now there exists the 0 eigenvalue. In the appendix, we argue on the validity of this result and on the explanation of the behavior shown in figure 1(*b*). Here it is enough to mention the fact that $\langle \mathcal{B} \rangle \leqslant 2$, for values of r large enough, is due to the appearance of the 0 eigenvalue.

3.2. Matrix representation and new operators

Further information on the violation of the CHSH inequality can be obtained by considering other representations of the pseudospin operator. Below we construct a new class of operators using the requirement that they fulfil the $SU(2)$ algebra.

An arbitrary operator can be expressed as

$$s_x = \sum_{n,m} s_{nm}^{(x)} |n\rangle\langle m|, \qquad s_y = \sum_{n,m} s_{nm}^{(y)} |n\rangle\langle m|, \qquad s_z = \sum_{n,m} s_{nm}^{(z)} |n\rangle\langle m|, \tag{26}$$

where $\langle n|s_i|m\rangle = s_{nm}^{(i)}$ are the matrix elements in the Fock basis. Hereafter, we will denote the matrix representation of the operator s_i by S_i. For example, the pseudospin operators have the following matrix representation in the Fock basis:

$$S_x = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = I \otimes \sigma_1, \qquad S_y = \mathrm{i} \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ -1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = -I \otimes \sigma_2,$$

$$S_z = \begin{pmatrix} -1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & -1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = -I \otimes \sigma_3.$$

Once the operators are written in this manner, it is clear that all their properties are inherited from the Pauli matrices. We can obtain further generalization by taking a right tensor product with an identity matrix, for example

$$S_{x,d} = I \otimes \sigma_x \otimes I_d, \qquad S_{y,d} = I \otimes \sigma_y \otimes I_d, \qquad S_{z,d} = I \otimes \sigma_z \otimes I_d,$$

$$S_{x,d} = \begin{pmatrix} 0 & I_d & \cdots \\ I_d & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \qquad S_{y,d} = \mathrm{i} \begin{pmatrix} 0 & -I_d & \cdots \\ I_d & 0 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}, \qquad S_{z,d} = \begin{pmatrix} I_d & 0 & \cdots \\ 0 & -I_d & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}. \tag{27}$$

Note that these matrices correspond to representations of the pseudospin operators $\vec{s}_d$ introduced by Larson [5].

We obtain new operators in the following way. Take two Hermitian matrices, A and B, such that $A^2 = I_A$ and $B^2 = I_B$, where I_B is the identity matrix with the same dimension as B and I_A is the identity matrix with the same dimension as A. With these matrices define

$$S'_x = I_B \otimes \sigma_x \otimes A, \qquad S'_y = B \otimes \sigma_y \otimes I_A, \qquad S'_z = B \otimes \sigma_z \otimes A. \tag{28}$$

It is straightforward to check that these operators satisfy the $SU(2)$ algebra and all of the Pauli matrices properties. For example, taking the matrices

$$A = \sigma_x, \qquad B = I_\infty, \tag{29}$$

the new operators become

$$S'_x = I_\infty \otimes \sigma_x \otimes \sigma_x, \qquad S'_y = I_\infty \otimes \sigma_y \otimes I_2, \qquad S'_z = I_\infty \otimes \sigma_z \otimes \sigma_x. \tag{30}$$

The explicit matrix representation of the operators is

$$S'_x = \begin{pmatrix} 0 & 0 & 0 & 1 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \qquad S'_y = \begin{pmatrix} 0 & 0 & -i & 0 & \cdots \\ 0 & 0 & 0 & -i & \cdots \\ i & 0 & 0 & 0 & \cdots \\ 0 & i & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

$$S'_z = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & -1 & \cdots \\ 0 & 0 & -1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

These pseudospin operators can also be expressed in the Fock bracket basis:

$$s'_x = \sum_{n=0}^{\infty} |4n\rangle\langle 4n+3| + |4n+1\rangle\langle 4n+2| + |4n+2\rangle\langle 4n+1| + |4n+3\rangle\langle 4n|, \tag{31a}$$

$$s'_y = -\mathrm{i}\sum_{n=0}^{\infty} |4n\rangle\langle 4n+2| + |4n+1\rangle\langle 4n+3| + \mathrm{i}|4n+2\rangle\langle 4n| + \mathrm{i}|4n+3\rangle\langle 4n+1|, \tag{31b}$$

$$s'_z = \sum_{n=0}^{\infty} |4n\rangle\langle 4n+1| + |4n+1\rangle\langle 4n| - |4n+2\rangle\langle 4n+3| - |4n+3\rangle\langle 4n+2|. \tag{31c}$$

Given these new operators, we can test their locality. To this end, we just calculate the I and F values using the $|\mathrm{TMSV}\rangle$ state:

$$F = \langle \mathrm{TMSV}| s'_x \otimes s'_x |\mathrm{TMSV}\rangle = \frac{4T^3}{(1+T^2)(1+T^4)}, \tag{32}$$

$$I = \langle \mathrm{TMSV}| s'_z \otimes s'_z |\mathrm{TMSV}\rangle = \frac{2T}{1+T^2}. \tag{33}$$

These quantities lead us to a maximum Bell expectation value given by

$$\langle B\rangle_{\max} = \frac{4T}{(1+T^2)(1+T^4)}\sqrt{6T^4+T^8+1}. \tag{34}$$

In figure 2, we plot $\langle B\rangle_{\max}$ as a function of the squeezing parameter r. We observe that violation of the CHSH inequality occurs only for squeezing parameters in the range $r > 0.6493$. This is in contrast with Chen's and Larson's pseudospin operators where the CHSH inequality is violated for all values of the squeezing parameter.

Finally, it is worth noticing that we can construct new pseudospin operators along the lines described in this section. According to our results, we expect different violation of the CHSH inequality for each of these operators.

4. Summary and conclusions

The pseudospin operator is an alternative way to study the entanglement of the TMSV state. We point out two possible drawbacks of the analogy between the pseudospin and the spin

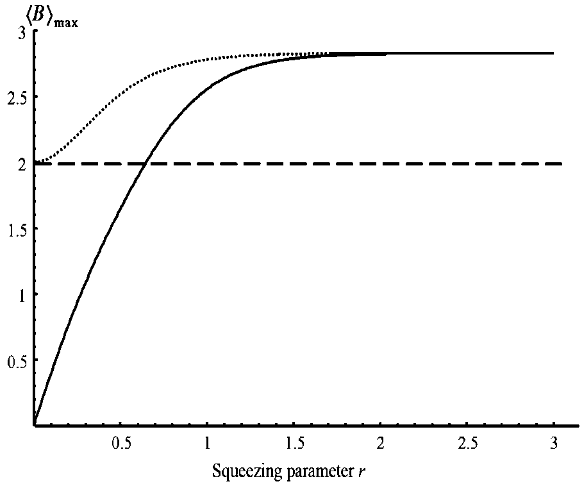

Figure 2. The expectation value of Bell's operator built in terms of the new pseudospin operators introduced in this paper (equation (32)) between TMSV states, as a function of the squeezing parameter r. The dotted line corresponds to the pseudospin case. The dashed line corresponds to the local realistic limit.

operators. One is the infinite degeneracy of the pseudospin and the other is a possible representation dependence of the CHSH inequality. We can summarize our findings as follows.

- Working with the state vectors and the operators built from the vector basis of a finite-dimensional Hilbert space, we have shown that the violation of the CHSH inequality is independent of the degeneracy of the pseudospin operator.
- We introduced an approach which allows us to construct generalized pseudospin operators. We recover as particular cases Chen *et al*'s and Larson's d-dependent operators.
- Keeping the $SU(2)$ algebra of the operators, but using a new representation, we found that the violation of the CHSH inequality is different from the original pseudospin, pointing thus to a representation-dependent violation of the CHSH inequality.
- We also considered the case in which the operator is defined over a finite-dimensional Hilbert space but the expectation value is taken over the full TMSV state. The interpretation in this case is obscured by the appearance of a zero eigenvalue which is responsible for producing $\mathcal{B} \leqslant 2$ for large enough values of the squeezing parameter r.

As far as the physical interpretation of our results is concerned, we must keep in mind the following points: (1) the degree of squeezing (r dependence) can be controlled in the laboratory, (2) the $r \rightarrow \infty$ limit amounts to considering the EPR state, so in that limit the prediction of all sensible models should agree, (3) experimentally it is not possible to work with a reduced Hilbert space chosen at will; therefore that point of our argument should be considered a mathematical trick. We consider that the degeneracy independence of the Bell inequality violation should be a prerequisite of the whole approach. On the other hand, the representation dependence suggests that the mapping between the continuous variable and the spin-1/2 system is not good enough.

Acknowledgments

We thank one of the referees for bringing this point to our attention.

Appendix. Zero eigenvalue

The definition of the truncated operators in equation (22) implies the appearance of eigenkets with zero eigenvalue. Indeed for $m \geqslant d(2L+2)$:

$$s_{x,d}^{(L)}|m\rangle = s_{y,d}^{(L)}|m\rangle = s_{z,d}^{(L)}|m\rangle = 0. \tag{A.1}$$

However, the Bell inequality remains valid. If instead of $\{-1, 1\}$ we take the set $\{-1, 1, 0\}$, i.e. if $A, A', B, B' \in \{-1, 1, 0\}$, it can be easily shown that

$$-2 \leqslant \langle AB\rangle + \langle A'B\rangle + \langle AB'\rangle - \langle A'B'\rangle \leqslant 2. \tag{A.2}$$

With the choice

$$A = \hat{a}\cdot\vec{s}_1,\ A' = \hat{a}'\cdot\vec{s}_1, \qquad B = \hat{b}\cdot\vec{s}_2,\ B' = \hat{b}'\cdot\vec{s}_2, \tag{A.3}$$

due to the correlations of the two modes in the TMSV state, no matter how we select the axes configuration, the outcomes $(0, 1), (0, -1), (1, 0), (-1, 0)$ will never occur. In fact, if we obtain 0 for a measure over an axis in one mode, then for the other mode we will also obtain 0 independently of the election of the axis. Consequently, for this case we have

$$\langle AB\rangle = \langle A'B\rangle = \langle AB'\rangle = \langle A'B'\rangle = 0. \tag{A.4}$$

For the TMSV, measuring both fields in the same axis, the only outcomes allowed are $(-1, 1), (1, -1), (1, 1), (-1, -1)$ and $(0, 0)$; therefore, there still exists a perfect correlation.

From the expression of the TMSV (12) it follows that each mode will have m photons with a probability $P_m = \frac{\tanh^m r}{\cosh r}$. For small r the first terms will correspond to the bigger probabilities. However, since $P_{n+1} = \tanh r\, P_n$ then when $r \to \infty$, $P_{n+1} \to P_n$, i.e. the probabilities tend to be same. Thus, all the kets in the expansion of the state will have the same probabilistic weight. The terms corresponding to $n \geqslant d(2L+2)$ have a vanishing contribution to the Bell operator, for $r \gg 1$ this produces a decrement in the mean value $\langle\mathcal{B}\rangle$. This is a qualitative explanation of why in figure 1(*b*) $\langle\mathcal{B}\rangle \to 0$ as $r \to \infty$.

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